Contents

1. Problem Formulation
   - Standard and Unknown Boundary Value Problems.
   - One and Two Phase Stefan Problems.

2. Numerical Solution of Moving Boundary Problems
   - Combination of Numerical Methods for Solving 1D problems with Moving Boundary.
   - The Algorithm and Its Application (Numerical Experiments).

3. Properties of Algorithm
   - Properties of the Solutions of the Riccati Equations.
   - Development on the One-Phase Stefan Problem.
   - Existence and Uniqueness of the Solution.
One-Phase Stefan Problem

The domain $\Omega \subset \mathbb{R}^2$, $\Omega \equiv \{(x, t) | 0 < x < s(t), 0 < t < \infty\}$ and $\Gamma \equiv \{(s(t), t) | 0 \leq t < \infty\}$. Find $\{u(x, t), s(t)\}$, where $u \in C^{(2,1)}$ and $s \in C^{(1)}$, such that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{for all} \quad (x, t) \in \Omega, \quad (1.4)$$

$$s(0) = 0, \quad u(0, t) = -1, \quad u(s(t), t) = 0, \quad t > 0, \quad (1.5)$$

$$\frac{\partial u}{\partial x} \bigg|_{x=s(t)} = L \frac{ds(t)}{dt}, \quad t > 0, \quad (1.6)$$

Figure 2.2.1. Physical realization of the one-phase Stefan problem.
Numerical Solution of MBPs

**A Mathematical Problem.** Find the triple \( \{u_1(x, t), u_2(x, t), s(t)\} \), for which

\[
L_i u_i \equiv \left( \frac{\partial}{\partial x} \left( p_i(x, t) \frac{\partial}{\partial x} \right) + a_i(x, t) \frac{\partial}{\partial x} - b_i(x, t) - d_i(x, t) \frac{\partial}{\partial t} \right) u_i = f_i(x, t),
\]

\( (x, t) \in \Omega_i, \quad t > 0, \quad i = 1, 2, \)

(2.1)

where \( \Omega_i \) is the subset of the rectangle \((l_1, l_2) \times (0, K)\) such that

\[
(x, t) \in \Omega_1 \iff t \in (0, K) \land x \in (l_1, s(t)) \equiv Q_1(s(t)),
\]

and

\[
(x, t) \in \Omega_2 \iff t \in (0, K) \land x \in (s(t), l_2) \equiv Q_2(s(t)),
\]

\( s(t) \) is the moving boundary, and \( K < \infty \) is some arbitrary but fixed upper limit.
Fixed boundary and initial conditions are,

\[ \alpha_i(t)u_i(l_i, t) + (-1)^i \beta_i(t)p_i(l_i, t) \frac{\partial u_i}{\partial x}(l_i, t) = \gamma_i(t), \quad t > 0, i = 1, 2 \quad (2.2) \]

\[ s(0) = s^0, \quad l_1 \leq s^0 \leq l_2, \quad (2.3) \]

\[ u_1(x, 0) = u_1^0(x), \quad l_1 \leq x \leq s^0, \quad u_2(x, 0) = u_2^0(x), \quad s^0 \leq x \leq l_2. \quad (2.4) \]

where, \( \alpha_i, \beta_i, \gamma_i, u_i^0, i = 1, 2 \), are given functions. We suppose that \( \alpha_i(t) \geq 0, \beta_i(t) \geq 0, \alpha_i(t) + \beta_i(t) \neq 0, i = 1, 2, t > 0. \)

Conditions at the moving interface \( x = s(t) \) will be written as

\[ H(u_1(s, t), u_2(s, t), p_1(s, t) \frac{\partial u_1}{\partial x}(s, t), p_2(s, t) \frac{\partial u_2}{\partial x}(s, t), s(t), \frac{ds(t)}{dt}, t) = 0, \quad t > 0, \quad (2.5) \]

where \( H = (H_1, H_2, H_3) \) is a given function with values in \( \mathbb{R}^3 \).
We first apply the method of time discretization called also Rothe’s method. Using implicit Euler discretization in time, the resulting approximate free boundary (interface) problem at the time level $t = t^n$ may be written as

\[
[p_i^n(x)u_i^{n'}]' + a_i^n(x)u_i^{n'} - b_i^n(x)u_i^n - d_i^n(x)\frac{u_i^n - \hat{u}_i^{n-1}(x)}{\Delta t} = f_i^n(x),
\]

\[x \in Q_i(s^n), \quad i = 1, 2, \tag{2.9}\]

\[
\alpha_i^n u_i^n(l_i) + (-1)^i \beta_i^n p_i^n(l_i) u_i^{n'}(l_i) = \gamma_i^n, \quad i = 1, 2, \tag{2.10}\]

\[
H^n(u_1^n(s^n), u_2^n(s^n), p_1^n(s^n)u_1^{n'}, p_2^n(s^n)u_2^{n'}, s^n, \frac{s^n - s^{n-1}}{\Delta t}, t^n) = 0, \tag{2.11}\]

Our task now is to find the triples $\{u_1(x), u_2(x), s\}$ satisfying (2.9)–(2.11) at successive times $t^n, n = 1, \ldots, N$. 

★ ★ ★ ★ ★
To solve the unknown interface problem at one time level we apply method of transfer of conditions by J. Taufer. For $\alpha_i > 0$ from the Theorem 2.1 (discussed in the paper) we solve

$$y_i^n' = \left[ \frac{d_i^n(x)}{\Delta t} + b_i^n(x) \right] y_i^n + \frac{a_i^n(x)}{p_i^n(x)} y_i - \frac{1}{p_i^n(x)} \quad a.e. \quad \text{on} \quad [l_1, l_2], \quad (2.14)$$

$$y_i(l_i) = (-1)^i \frac{\beta_i^n}{\alpha_i^n}, \quad i = 1, 2,$$

$$z_i^n' = \left[ \frac{d_i^n(x)}{\Delta t} + b_i^n(x) \right] y_i^n z_i^n - \left[ \frac{\hat{u}_i^{n-1}(x)}{\Delta t} - f_i^n(x) \right] y_i^n(x) \quad a.e. \quad \text{on} \quad [l_1, l_2], \quad (2.15)$$

$$z_i^n(l_i) = \frac{\gamma_i}{\alpha_i}, \quad i = 1, 2.$$

The functions $y_i, z_i$ possess the property that any absolutely continuous function $u_i$ that satisfies ODE a.e. and for which fixed boundary condition holds satisfies also the transferred condition.
Numerical Solution of MBPs

\[ u^n_i(x) = z^n_i(x) - y^n_i(x)p^n_i(x)u^n_i'(x) \quad \forall x \in [l_1, l_2], \quad i = 1, 2. \quad (2.16) \]

If \( y^n_i(x) \neq 0 \) we may express \( p^n_i(x)u^n_i' \) from (2.16) and substitute into the unknown interface condition:

\[ H^n \left[ u^n_1(s^n), u^n_2(s^n), \frac{z^n_1(s^n) - u^n_1(s^n)}{y^n_1(s^n)}, \frac{z^n_2(s^n) - u^n_2(s^n)}{y^n_2(s^n)}, s^n, s^n - s^{n-1}, \Delta t, t^n \right] = 0. \quad (2.23) \]

The transferred condition (2.16) is sometimes called \textit{Riccati transformation} since (2.14) is the \textit{Riccati equation}. 
Properties of Algorithm

We learn the properties of the algorithm and work on the following two problems:

- The feasibility of algorithm.
- The existence of the interface $s^n$.

An ordinary differential equation of the form

$$y'(x) = A(x)y^2(x) + B(x)y(x) + C(x)$$

is known as a **Riccati equation** or a **generalized Riccati equation**

Let $y$ and $\omega$ be such that $y(x) = \frac{\omega(x)}{e^{-\int_0^x B(\xi)d\xi}} = \frac{\omega(x)}{E(x)}$, $x \in [0, 1]$ then the **Lemma 3.1** (in the paper) enables us to study the following equation instead of (3.1)

$$\omega' = P(x)\omega^2 + Q(x), \quad \forall x \in (0, 1), \quad \omega(0) = y_0.$$  

In addition, $\text{sgn } P(x) = \text{sgn } A(x)$, $\text{sgn } Q(x) = \text{sgn } C(x)$, $\text{sgn } \omega(x) = \text{sgn } y(x)$.  

⭐⭐⭐
**Theorem 3.3.** Let $P, Q \in C([0, 1])$ and $P(x) \geq 0, Q(x) \leq 0 \ \forall x \in [0, 1]$. Let also $\omega_0 \leq 0$ and $\omega_0 + Q(0) < 0$. Then there exists a unique continuous function $\omega : [0, 1] \rightarrow R$ which satisfies the equation (3.4) on $[0, 1]$ with the initial condition $\omega(0) = \omega_0$ and furthermore $\omega(x) < 0 \ \forall x \in (0, 1]$.

★★★

Development on One-Phase Stefan Problem

\[
Lu \equiv u_{xx} + a(x, t)u_x - b(x, t)u - d(x, t)u_t = f(x, t), \quad (x, t) \in \Omega_0, \tag{3.18}
\]

where $\Omega_0 = \{(x, t) : 0 < x < s(t), t > 0\}$. The conditions at the moving interface are

\[
u(s(t), t) = 0, \quad t > 0, \tag{3.20a}
\]

\[
\frac{ds}{dt} + k(s(t), t)u_x(s(t), t) = \eta(s(t), t), \quad t > 0. \tag{3.20b}
\]
Properties of Algorithm

**Assumption 3.1**

1. All the functions in the equation (3.18) and in the condition (3.20b) are continuous and bounded on $[0, \infty) \times [0, K]$, the function $\alpha$ is continuous on $[0, K]$ and the function $u_0$ is continuous on $[0, s^0]$.

2. We suppose that $\alpha(t) > 0$, $t \in [0, K]$; $k(x, t) \geq 0$, $\eta(x, t) \geq 0$, $0 \leq a \leq a(x, t) \leq \bar{a}$, $0 \leq b \leq b(x, t) \leq \bar{b}$, $0 \leq d \leq d(x, t) \leq \bar{d}$, $f(x, t) \leq 0$, $(x, t) \in [0, \infty) \times [0, K]$; $u_0(x) \geq 0$, $x \in [0, s(0)]$, and $\alpha(0) = u_0(0)$.

3. There exist constants $M \geq 0$ and $\beta \geq 0$ such that

$$u_0(x) \leq M \frac{s^0 - x}{s^0}, \quad f(x, t) \geq bMe^{\beta t} \frac{x - s^0}{s^0}, \quad x \in [0, s^0], \quad t \in [0, K].$$

4. $f(x, t) = 0$, $x \geq s^0$, $t \in [0, K]$.

5. Functions $\eta(x, t)$, $k(x, t)$ are continuously differentiable on $[0, \infty) \times [0, K]$. 
Existence of Solution

Using the time discretization method with the time step $\Delta t$, $n = 1, 2, \ldots, N$, the approximate problem for (3.18)–(3.20) is

\[
L^n u^n \equiv u^{n''} + a^n(x) u^{n'} - [b^n(x) u^n + \frac{d^n(x)}{\Delta t}] u^n =
\]

\[
= f^n(x) - \frac{d^n(x)}{\Delta t} \hat{u}^{n-1}(x), \quad 0 < x < s^n, \tag{3.21}
\]

\[
u^n(0) = \alpha^n, \tag{3.22}
\]

\[
u^n(s^n) = 0, \quad s^n - s^{n-1} + \Delta t k^n(s^n) u^{n'}(s^n) = \Delta t \eta^n(s^n). \tag{3.23}
\]

**Theorem 3.5.** Let $u^{n-1}$, $s^{n-1}$ be given and $u^{n-1}(x) \geq 0$, $x \in [0, s^{n-1}]$. If the problem (3.21)–(3.23) has a solution $\{u^n, s^n\}$, then $u^n(x) \geq 0$ for $x \in [0, s^n]$ and $s^n \geq s^{n-1}$.
**Theorem 3.6.** The free boundary \( s^n \) of (3.21)–(3.23) at \( t = t^n \), if it exists, is a root of the equation \( \varphi^n(x) = 0 \). Conversely, any zero of \( \varphi^n \) in \( (0, \infty) \) represents an admissible free boundary \( s^n \) of the problem (3.21)–(3.23).

\[
\varphi^n(x) = x - s^{n-1} - \Delta t \left[ \eta^n(x) - k^n(x) \frac{z^n(x)}{y(x)} \right].
\]  

\hspace{1cm} (3.28)

**Theorem 3.7.** Given a couple \( \{u^{n-1}, s^{n-1}\} \) such that \( s^{n-1} \geq s^0 \), \( u^{n-1}(x) \geq 0 \) for \( x \in [0, s^{n-1}] \), there exists a solution \( \{u^n, s^n\} \) of the free boundary problem (3.21)–(3.23) and any such solution satisfies \( u^n(x) \geq 0 \), \( x \in [0, s^n] \), and \( s^n \geq s^{n-1} \).
Lemma 3.2. Let $M$ and $\beta$ be the constants from Assumption 3.1-3. Put

$$\|\alpha\| = \max_{[0,K]} |\alpha(t)|, \quad M_1 = \max(M, \|\alpha\|).$$

Suppose that

$$0 \leq u^{n-1}(x) \leq M_1 \frac{s^{n-1} - x}{s^{n-1}} e^{\beta(n-1)\Delta t}, \quad x \in [0, s^{n-1}].$$

Then we have

$$0 \leq u^n(x) \leq M_1 \frac{s^n - x}{s^n} e^{\beta n \Delta t}, \quad x \in [0, s^n]$$

for every solution $u^n, s^n$. 
Uniqueness of Solution

Theorem 3.8. Let $\|\eta\|$, $\|k\|$ be constants such that $|\eta(x, t)| \leq \|\eta\|$, $|k(x, t)| \leq \|k\|$, $x \in [0, \infty)$, $t \in [0, K]$. Let $\beta$ be the constant from Assumption 3.1-3 and let $M_1$ be the constant from Lemma 3.2. Set

$$\hat{M}_1 = M_1 e^{\beta K}, \quad C = \|\eta\| + \|k\| \frac{\hat{M}_1}{s^0}.$$

Then any solution $\{u^n, s^n\}$ of the problem (3.21)–(3.23) satisfies

$$0 \leq u^n(x) \leq \hat{M}_1, \quad x \in [0, s^n], n = 0, 1, \ldots, N,$$

(3.31)

$$0 \leq s^n - s^{n-1} \leq C \Delta t, n = 1, 2, \ldots, N,$$

(3.32)

and thus

$$0 \leq s^n \leq s^0 + C K, \quad n = 1, 2, \ldots, N.$$

(3.33)
Uniqueness of Solution

Theorem 3.9. For a sufficiently small $\Delta t$, the problem (3.21)–(3.23) has a unique solution $\{u^n, s^n\}$.

\[ \varphi' \geq 1 - \Delta t \|\eta_x\| - \|k_x\|\sqrt{\Delta t}e^{\bar{a}s^0} \sqrt{U} \left[ \tanh \left( \sqrt{\frac{U}{\Delta t}} s^0 \right) \right]^{-1} \cdot M_1 \quad (3.39) \]

for $x \in [s^{n-1}, s^{n-1} + C]$. Hence, for sufficiently small $\Delta t$ we obtain the estimate

\[ \varphi^n'(x) > 0, \quad x \in [s^{n-1}, s^{n-1} + C], \]

and thus the function $\varphi^n(x)$ is increasing on $[s^{n-1}, s^{n-1} + C]$ and has at most one zero.
Problem. A slab, $0 \leq x \leq 1$, initially solid at temperature $T_s = u_2(x, 0) = -0.5^\circ C$ and $u_1(x, 0) = 0^\circ C$ (just formally), $s(0) = 0$, is melted from the left by imposing a temperature $T_l = u_1(0, t) = 1^\circ C$ at the face $x = 0$ and at the back face $x = 1$ use the Neumann temperature itself. Find the triple $\{u_1(x, t), u_2(x, t), s(t)\}$, where $u_1$ and $u_2$ denote the temperature in the liquid and solid phases, respectively.

Location of the free boundary $s(t)$ computed by two different algorithms

<table>
<thead>
<tr>
<th>$t$</th>
<th>Alg1</th>
<th>RE1</th>
<th>Alg2</th>
<th>RE2</th>
<th>True Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0309</td>
<td>0.0351</td>
<td>0.0140</td>
<td>0.0346</td>
<td>0.0281</td>
<td>0.0356</td>
</tr>
<tr>
<td>6.0460</td>
<td>0.0461</td>
<td>0.0573</td>
<td>0.0447</td>
<td>0.0252</td>
<td>0.0436</td>
</tr>
<tr>
<td>8.0470</td>
<td>0.0523</td>
<td>0.0398</td>
<td>0.0505</td>
<td>0.0040</td>
<td>0.0503</td>
</tr>
</tbody>
</table>

★★★★
Numerical Solution of Temperature Distribution of the 2-phase Stefan problem with Alg1

at t=4.0309

at t=6.0460

at t=8.0470
The Numerical Experiments

Graph of Moving Interface at three times

Numerical Solution with Alg1
Numerical Solution of Temperature Distribution of the 2-phase Stefan problem with Alg2

- at $t=4.0309$
- at $t=6.0460$
- at $t=8.0470$
The Numerical Experiments

Graph of Moving Interface at three times $t$ with $s(t)$.

- Numerical Solution with Alg2
The Numerical Experiments

Neumann Solution of Temperature Distribution of the 2–phase Stefan problem

Numerical Solution of Stefan-Like Problems, Theory and Algorithm – p. 22/26
The Numerical Experiments

Graph of Moving Interface at three times

$t$(s)
The Numerical Experiments

Neumann Solution of Temperature Distribution of the 2-phase Stefan problem

$u(x)$

at $t=4.0309$
The Numerical Experiments

Neumann Solution of Temperature Distribution of the 2-phase Stefan problem

u(x)

at t=4.0309
at t=6.0460

Numerical Solution of Stefan-Like Problems, Theory and Algorithm – p. 25/26
The Numerical Experiments

Neumann Solution of Temperature Distribution of the 2-phase Stefan problem

Numerical Solution of Stefan-Like Problems, Theory and Algorithm – p. 26/26